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## On sampling numbers in $L_2$

Let $F \subset L_2$ be a class of complex-valued functions on a set $D$, such that, for all $x \in D$, point evaluation $f \mapsto f(x)$ is a continuous linear functional. We study the $L_2$-approximation of functions from $F$ and want to compare the power of function values with the power of arbitrary linear information. To be precise, the *sampling number* $g_n(F)$ is the minimal worst-case error (in $F$) that can be achieved with $n$ function values, whereas the *approximation number* (or Kolmogorov width) $d_n(F)$ is the minimal worst-case error that can be achieved with $n$ pieces of arbitrary linear information (like derivative values or Fourier coefficients).
Here, we report on recent developments in this problem and, in particular, explain how the individual contributions from [1-4] lead to the following statement:
There is a universal constant $c \in \mathbb{N}$ such that the sampling numbers of the unit ball $F$ of every separable reproducing kernel Hilbert space are bounded by

$$g_{cn}(F) \leq \sqrt{\frac{1}{n} \sum_{k \geq n} d_k(F)^2}.$$

We also obtain similar upper bounds for more general classes $F$, and provide examples where our bounds are attained up to a constant. For example, if we assume that $d_n(F) \asymp n^{-\alpha}(\log n)^{\beta}$ for some $\alpha > 1/2$ and $\beta \in \mathbb{R}$, then we obtain

$$g_n(F) \asymp d_n(F),$$

showing that function values are (up to constants) as powerful as arbitrary linear information.
The results rely on the solution to the Kadison-Singer problem, which we extend to the subsampling of a sum of infinite rank-one matrices.

**References.**

[1] M. Dolbeault, D. Krieg and M. Ullrich, A sharp upper bound for sampling numbers in $L_2$, preprint.

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[3] D. Krieg and M. Ullrich, Function values are enough for $L_2$-approximation: Part II, *J. Complexity* **66** (2021).

[4] N. Nagel, M. Schäfer and T. Ullrich, A new upper bound for sampling numbers, *Found. Comput. Math.* **21** (2021).