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An algebraic characterization of B-splines

B-splines of order k can be viewed as a mapping N taking a $(k+1)$-tuple of increasing real numbers $a_0 < \cdots < a_k$ and giving as a result a certain piecewise polynomial function. Looking at this mapping N as a whole, basic properties of B-spline functions imply that it has the following algebraic properties: (1) $N(a_0, \ldots, a_k)$ has local support contained in the interval $[a_0, a_k]$; (2) $N(a_0, \ldots, a_k)$ allows refinement, i.e. for every $a \in \cup_{j=0}^{k-1}(a_j, a_{j+1})$ we have that if $(\alpha_0, \ldots, \alpha_{k+1})$ is the increasing rearrangement of the points $\{a_0, \ldots, a_k, a\}$, the 'old' function $N(a_0, \ldots, a_k)$ is a linear combination of the 'new' functions $N(\alpha_0, \ldots, \alpha_k)$ and $N(\alpha_1, \ldots, \alpha_{k+1})$; (3) N is translation and dilation invariant. It is easy to see that derivatives of $N(a_0, \ldots, a_k)$ satisfy properties (1)–(3) as well.
Let F be a mapping taking $(k+1)$-tuples of increasing real numbers to some generalized function. In this paper we show that under some additional mild condition on the size of the supports of $F(a_0, \ldots, a_k)$ relative to the interval $[a_0, a_k]$, properties (1)–(3) are already sufficient to deduce that $F(a_0, \ldots, a_k)$ is a non-zero multiple of (some derivative of) a B-spline function. However, and somewhat surprisingly, we explicitly give examples of choices of F satisfying (1)–(3) but are not of this form.

This is a joint work with Anna Kamont.