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Stable nonlinear manifold and Lipschitz widths

Dozens of widths are known, see e.g. [1]. Generally they are sequences of numbers that measure the best, *possible under given restrictions,* approximation of the set $\mathcal{K} \subset X$ where $\mathcal{K}$ is (usually) a compact subset of a Banach space X.
In recent decades in numerical analysis we see the growing interest in non-linear algorithms. While it is well known that nonlinear methods of approximation can often perform dramatically better than linear methods, there are still questions on how to measure the optimal performance possible for such methods. Some attempts were made in [2] however they were not taking into account the numerical stability of the methods. In the talk I present two types of widths taking this into account.
In [3] we introduce stable manifold width

$$\delta_{n,\gamma}(\mathcal{K}) = \inf_{a,M,\|.\|_Y} \sup_{k \in \mathcal{K}} \|f - M(a(k))\|_X$$

where $a : \mathcal{K} \to \mathbb{R}^n$, $M : \mathbb{R}^n \to X$, $\|.\|_Y$ is a norm on $\mathbb{R}^n$ and both a and M are γ-Lipschitz map when on $\mathbb{R}^n$ we use $\|.\|_Y$. In [4] we introduce Lipschitz widths

$$d_n^{\gamma}(\mathcal{K}) = \inf_{\|\cdot\|_Y} \inf_{\Phi_n} \sup_{k \in \mathcal{K}} \inf_{y \in B_n} \|k - \Phi_n(y)\|_X$$

where $\|.\|_Y$ is a norm on $\mathbb{R}^n$, B_n is a unit ball in $\mathbb{R}^n$ equipped with norm $\|.\|_Y$ and $\Phi_n : B_n \to X$ is a γ-Lipschitz map.

In the talk I will try to present

1. The justification for such widths.
2. Basic properties of those new widths and their relations with Banach spaces.
3. Relations between new withs and classical ones especially the entropy numbers.
4. Present some results and examples motivated by important various numerical procedures specially by deep learning.

References.

[1] A. Pinkus, n-widths in Approximation Theory, Springer, 2012

[2] R. DeVore, R. Howard, C. Micchelli, *Optimal nonlinear approximation*, Manuscripta Mathematica **63**(4) (1989), 469-478

[3] A. Cohen, R. DeVore, G. Petrova, P. W. *Optimal stable nonlinear approximation*, Found. Comput. Math. 22 (2022), no. 3, 607-648

[4] G. Petrova, P. W. *Lipschitz widths*, Constructive Approximation (to appear)